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Rapid-distortion turbulence models in the theory of surface-wave generation

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Turbulent air flow over a surface gravity wave of small amplitude is studied analytically on the basis of a family of rapid-distortion turbulence models. Results for the wave growth rate do not depend sensitively on the specific choice of these models. However, the agreement with results based on a so-called truncated mixing-length model (Belcher & Hunt 1993) is poor, despite physical similarity of the models. The present analysis also shows that the use of turbulence models based on rapid-distortion theory leads to significant underestimation of observed growth rates of high-frequency waves.

1. Introduction

In the theory of generation of surface gravity waves by turbulent air flow a critical question is the choice of turbulence model used to describe the interaction of the wave and the turbulence. Many such models, of varying complexity, have been developed during the past twenty-five years, see e.g. Miles (1993) for a brief review.

Recently, Belcher & Hunt (1993, hereinafter referred to as BH) introduced a turbulence model based on results of rapid-distortion theory (Britter, Hunt & Richards 1981), which differs essentially from the previous models. According to BH, the classical eddy viscosity model (and modifications thereof), applied throughout the domain of air flow, are inappropriate because the wave-induced Reynolds stresses in the region away from the water surface (the so-called outer layer) are not properly described.

In §2 we summarize the Belcher–Hunt model, which implies truncation of the mixing length in the outer layer. Because of the discontinuity in this model, we propose a modified version which is continuous throughout the flow domain but is otherwise based on rapid-distortion theory. To study the sensitivity of the wave growth rate to the specific choice of turbulence model, we introduce a family of closure schemes, where the magnitude of the stresses in the outer layer may be varied substantially, while the stresses in the layer close to the water surface remain almost unchanged.

In §3 this family of closure schemes is studied analytically, which leads to closed-form expressions for the wave growth rate in terms of the model parameters. The results thus obtained are compared with those derived by BH. In addition, comparisons are made with observed growth rates (§4). For details of the calculations the reader is referred to van Duin (1996, hereinafter referred to as I) and to an internal memorandum (van Duin 1994).

2. The turbulence models

We start from the linearized Reynolds equations for air, written as

$$U\frac{\partial u}{\partial x}+\frac{\mathrm{d}U}{\mathrm{d}z}v = -\frac{\partial p}{\partial x}+\epsilon\frac{\partial \sigma_{11}}{\partial x}+\epsilon\frac{\partial \sigma_{12}}{\partial z}, \tag{2.1}$$

$$U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial z}+\epsilon\frac{\partial \sigma_{21}}{\partial x}+\epsilon\frac{\partial \sigma_{22}}{\partial z}. \tag{2.2}$$

In these dimensionless equations the coordinates are scaled by $L = 1/k$ (where k is the horizontal wavenumber), and the velocities by V (where V is the wind speed at the reference height L). The pressure p was scaled by ρV^2 (with ρ the constant air density), and the wave-induced Reynolds stresses by $\rho u_* V$ (with u_* the friction velocity). The parameter ϵ is defined by $\epsilon = u_*/V$ and is assumed small. The horizontal x-axis is aligned with the unidirectional mean flow U, which varies with height z. The (x, z) reference frame moves with the phase velocity of the wave.

It will be convenient to introduce the transformation $z = y+\eta_w(x, t)$, where η_w is the surface elevation, and y is the height above the moving water surface. The stream function ϕ, defined in the orthogonal coordinate system, is written as $\phi(x, z, t) = \bar{\psi}(y)+\psi(x, y, t)$, where $\bar{\psi}$ is the x-averaged part (for fixed y), and ψ is the perturbation part; cf. I.

In the so-called intermediate layer $y = O(\epsilon)$, where the turbulence is supposed to be in a local equilibrium, BH introduce the mixing-length model

$$\sigma_{12} = -2\kappa y\psi_{yy}, \quad \sigma_{11} = -e_1\sigma_{12}, \quad \sigma_{22} = -e_2\sigma_{12}, \tag{2.3}$$

rewritten in the present notation, where $-\psi_y$ is the horizontal perturbation velocity in the displaced coordinate system. In this turbulence model of Townsend (1976) the normal stresses are proportional to the shear stress, where e_1 and e_2 are positive model constants.

According to rapid-distortion theory, the mixing-length model ceases to be valid in the outer layer $y = O(1)$. For this reason, BH introduce a so-called truncated mixing-length model, which implies that the mixing length is truncated in this layer.

The discontinuity in the truncated mixing-length model (or Belcher–Hunt model) leads to ambiguous results (§3). Therefore, we introduce the following modification:

$$\sigma_{12} = -2\kappa y[1+(hy/\epsilon^{1/2})^{2n}]^{-1}\psi_{yy}, \quad \sigma_{11} = -e_1\sigma_{12}, \quad \sigma_{22} = -e_2\sigma_{12}, \tag{2.4}$$

which is valid throughout the domain of air flow. Here h is a model constant of the order of unity, and n is a positive integer, which serves to investigate the dependence of the wave growth rate on the choice of turbulence model. As will be shown, the models are identical in the intermediate layer. In the outer layer we have $\sigma_{ij} = O(\epsilon^n ak)$, where ak is the wave slope. This order of magnitude is in agreement with rapid-distortion theory; cf. formula (3.4) in BH. For large n the outer layer is inviscid, as in the Belcher–Hunt model.

3. Analysis of the turbulence models

In referring to I for details of the calculations, it will be convenient to write the water displacement as $\eta_w = \epsilon A \exp(\mathrm{i}x)$, with $A = O(1)$. This corresponds to perturbation quantities that are $O(\epsilon)$. Since the present theory is linear, the factor $A\exp(\mathrm{i}x)$ will be divided out.

The stream function in the outer layer is written as $\phi(x,z,t) = \bar{\phi}(z) + \tilde{\phi}(x,z,t)$, where $\bar{\phi}$ is the x-averaged part (for fixed z), and $\tilde{\phi}$ is the perturbation part. Then (2.1) and (2.2) reduce to the single equation

$$\bar{\phi}_z \nabla^2 \tilde{\phi}_x - \bar{\phi}_{zzz} \tilde{\phi}_x = \epsilon \frac{\partial^2(\sigma_{11}-\sigma_{22})}{\partial z\, \partial x} + \epsilon\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2}\right)\sigma_{12}. \tag{3.1}$$

The above decomposition is expanded according to

$$\bar{\phi}(z) = \bar{\theta}_0 + \epsilon\bar{\theta}_1, \quad \tilde{\phi}(x,z,t) = \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \dots . \tag{3.2}$$

The expressions for $\bar{\theta}_0$ and θ_1 are given by (A 1) in the Appendix; cf. I(4.8) (i.e. equation (4.8) in I) with $Q = 0$ and $\alpha = 1$, where c denotes the dimensionless phase velocity. The perturbation part determines the Reynolds stresses according to (2.4). For the shear stress in the outer layer we obtain

$$\sigma_{12} = -2\kappa z \epsilon^{1+n} (hz)^{-2n}\, \theta_{1zz} + \dots . \tag{3.3}$$

In view of (3.3) the right-hand side of equation (3.1) is $O(\epsilon^3)$, which implies that the expressions for θ_1 and θ_2 in (3.2) are the same as in I; cf. (A 2) and I(4.7), I(4.9).

Since we are mainly interested in the determination of the wave growth rate, only the imaginary part of the pressure needs be considered (I). Then it will be convenient to introduce the notation $z_r = \mathrm{i}\,\mathrm{Im}\, z$ for any complex z, and z_r is called the reduced (part of) z.

The equation for θ_3 has a reduced solution of the form (A 3), where h_3 is a real constant, and δ_{n1} is the Kronecker symbol. With (A 1) and (A 2) this determines the reduced pressure in the outer layer to $O(\epsilon^3)$. When expressed in terms of the transition-layer variable ζ, defined by $\zeta = y/\epsilon^{1/2}$, we obtain

$$p_r \to \epsilon^3 \log(1/\epsilon)\left(\frac{2\mathrm{i}\kappa w}{h^2}\right)\delta_{n1} - \mathrm{i}\epsilon^3 h_3\, w + \epsilon^3\left[\frac{-4\mathrm{i}\kappa w}{h^2}\log\zeta - \frac{2\mathrm{i}\kappa w}{h^2}(\gamma + \log 2)\right]\delta_{n1} \quad \text{as} \quad y \to 0. \tag{3.4}$$

In the transition layer, $\zeta = O(1)$, the perturbation stream function is written as

$$\psi_{tl} = \epsilon^{3/2}\psi_1(\zeta) + \epsilon^2\psi_2(\zeta) + (\epsilon^{5/2}\log\epsilon)\,\psi_3(\zeta) + \epsilon^{5/2}\psi_4(\zeta) + (\epsilon^3\log\epsilon)\,\psi_5(\zeta) + \epsilon^3\psi_6(\zeta) + \dots . \tag{3.5}$$

In deriving the equations for the various $\psi_i(\zeta)$, it turns out that ψ_1, ψ_2, ψ_3 and ψ_5 are in fact solutions of Rayleigh's equation. The equations for ψ_4 and ψ_6, on the other hand, are modified by the Reynolds stresses. The solutions are given by (A 4) and (A 5). The reduced pressure gradient in the transition layer is of the form

$$\frac{\mathrm{d}p_r}{\mathrm{d}\zeta} = \epsilon^3\left[\frac{-4\mathrm{i}\kappa w\zeta}{1+(h\zeta)^{2n}} + 2\kappa e_2 \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\left(\frac{\zeta}{1+(h\zeta)^{2n}}\right)\psi_{4r\zeta\zeta}\right)\right] + \dots . \tag{3.6}$$

In the intermediate layer, $\eta = y/\epsilon$, the perturbation stream function is written as

$$\psi_{im} = \epsilon^2\varphi_2(\eta) + (\epsilon^3\log\epsilon)\,\varphi_3(\eta) + \epsilon^3\varphi_4(\eta) + \dots . \tag{3.7}$$

The equations for φ_2 and φ_3 have solutions of the form (A 6*a*, *b*) and are the same as in I; cf. I(5.8) and I(5.9). The equation for φ_4, given by I(5.4), has a solution of the form (A 6*c*), where $\omega_3(\eta)$ satisfies (A 7). The reduced pressure gradient reads

$$\frac{\mathrm{d}p_r}{\mathrm{d}\eta} = \epsilon^3 \frac{\mathrm{d}}{\mathrm{d}\eta}[\mathrm{i}we_2\,\alpha_2\,\omega_3(\eta)] + \dots . \tag{3.8}$$

The lower-order solutions are independent of any closure assumptions. For small ϵ and slow waves (the conditions for validity of the present analysis) it is then expected that the predicted vertical-velocity profiles agree with measurements. The results in van Duin & Janssen (1992) seem to confirm this.

The pressure in the intermediate layer is of the form

$$p_{im} = -\epsilon w^2 + \epsilon^2\left(w^2\eta + \frac{2w}{\kappa} + w(\alpha_1 + 2\mathrm{i}\kappa)\right) + \dots, \tag{3.9}$$

with $\alpha_{1r} = -2\mathrm{i}\kappa$, determined by matching (3.7) with (3.5). This implies that $p_r = o(\epsilon^2)$ in the intermediate layer. Thus, in order to find the expression for the imaginary part of the pressure in this layer, which determines the wave growth rate, the higher-order terms in (3.9) should be known. With (3.4), (3.6) and (3.8) we obtain

$$p_r = (\epsilon^3 \log(1/\epsilon))\frac{2\mathrm{i}\kappa w}{h^2}\delta_{n1} + \epsilon^3\left[-\mathrm{i}wh_3 + \mathrm{i}we_2\alpha_2\omega_3(\eta)\right.$$
$$\left. + \frac{2\mathrm{i}\kappa w}{h^2}(\log(h^2/2) - \gamma)\delta_{n1} + \frac{2\mathrm{i}\kappa w}{h^2}(1-\delta_{n1})\int_0^\infty \frac{\mathrm{d}t}{1+t^n}\right] + \dots, \tag{3.10}$$

where the constants h_3 and α_2 are still to be determined (§4).

The truncated mixing-length model (2.3) corresponds to the model (2.4) for $n = 0$, where in (2.3) the outer layer is inviscid by construction. For reasons mentioned below (3.3), the outer solution (3.2) then remains of the form (A 2), (A 3). The expression (3.4) for the reduced outer-layer pressure also remains the same. The transition layer needs not be taken into account in this case because the intermediate-layer solution can now be matched directly with the outer solution. Then the former solution remains of the form (A 6), where matching leads to $\alpha_{1r} = 0$ because of the real expressions (A 2). This implies that the reduced intermediate-layer pressure is $2\kappa w\epsilon^2$ at leading order, in view of (3.9). On the other hand, (3.4) implies that the reduced outer-layer pressure is $O(\epsilon^3)$.

Based on this result we conclude that the outer-layer pressure cannot be matched with the intermediate-layer pressure because the imaginary parts are of different order, which implies a discontinuity in the shear-stress gradient across the outer and intermediate layers.

4. The wave growth rate

The growth rate coefficient is defined according to

$$(kcE)^{-1}\partial E/\partial t = s(u_*/c)^2\beta, \tag{4.1}$$

where E is the energy of the inviscid water wave and s is the air–water density ratio.

To determine the growth rate coefficient, the constants h_3 and α_2 in (3.10) should be known. Matching the solution I(3.1), I(3.7) in the so-called inner layer (where molecular viscosity may not be neglected) with (3.7) yields $\alpha_2 = 4\mathrm{i}$. Matching (3.7) via (3.5) with the outer solution, we obtain $h_3 = -4 + 2/w + (2\kappa/h^2)(\gamma + \log 2)\delta_{n1}$. Matching the normal stress with the pressure at the surface elevation (I) then leads to the leading-order expressions

$$\beta = \log(1/\epsilon)\frac{2\kappa w}{h^2} + \dots \quad (n = 1), \tag{4.2}$$

$$\beta = -2 + 4w + \frac{2\kappa w}{h^2}\int_0^\infty \frac{\mathrm{d}t}{1+t^n} + \dots \quad (n = 2, 3, 4, \dots). \tag{4.3}$$

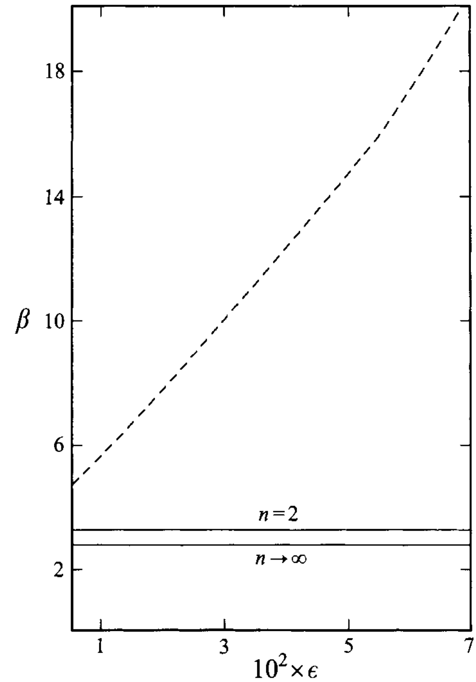

FIGURE 1. Variation of the growth rate coefficient β with the parameter ϵ (for slow waves). Solid lines: the result (4.3) with $h=1$; dashed line: the result (4.4) obtained by Belcher & Hunt (1993).

Thus, the wave growth rate (4.2), (4.3) is asymptotically largest for $n=1$ and decreases with increasing n. This corresponds to decreasing magnitude of the outer-layer shear stress (3.3). However, a significant change in the magnitude of this stress with varying n only leads to a relatively small change of the wave growth rate (see figure 1). The validity of rapid-distortion turbulence models is restricted to slow waves (BH), which corresponds to $w\approx 1$. Then the growth rate is largest for $n=1$ if $\epsilon<0.017$, which corresponds to unrealistically small drag coefficient.

Excluding the special case $n=1$, we conclude that the wave growth rate does not depend sensitively on the choice of rapid-distortion turbulence model. The reason for this is that, typical of such models, the interaction between the wave and the turbulence mainly takes place in the layers close to the water surface, where the turbulence is in a local equilibrium. In these layers, with $y=O(\epsilon)$, the turbulence model (2.4) actually reduces to Townsend's mixing-length model (2.3). This is also readily seen from the expression for the shear stress in the intermediate layer. The leading-order expression for this stress, given by $\sigma_{12}=-2\epsilon^2\kappa\eta\varphi_{4\eta\eta}$, is indeed independent of n because the constant α_2 in (A 6c) is independent of this parameter, see below (4.1). Only in the higher layers, where distortion of the eddies takes place, are the models (2.3) and (2.4) different.

For $c\ll 1$ (slow waves) BH obtain the result

$$\beta = 2(2W^4+W^2-1), \quad W=\frac{R+\log h}{R+\log \ell}, \quad R=-\log(kz_0), \tag{4.4}$$

with $h^2(R+\log h)=1$, $\ell(R+\log\ell)=0.32$, where z_0 is the roughness length; cf. Miles (1996).

In the limit $n\to\infty$ the model (2.4) and the Belcher–Hunt model are physically the same. Thus, it is expected that in this limit the wave growth rates (4.3) and (4.4) are of the same order of magnitude. It turns out, however, that smoothing the discontinuity leads to a significant reduction of the wave growth rate. This is depicted in figure 1, where $\epsilon = \kappa/R$ corresponds to the square root of the drag coefficient. Only for very small ϵ (or very smooth flow) is there some agreement.

Based on several field and laboratory experiments, Plant (1982) derived an empirical expression for the wave growth rate coefficient, which (due to scatter in the data) may vary between 20 and 60. When compared with the maximum value $\beta_m = 3.25$, obtained from (4.3) for $n = 2$, $h = 1$ and $c \downarrow 0$, we conclude that the predicted high-frequency wave growth rates are significantly smaller than the observed values. This also applies to a related second-order closure model of Launder, Reece & Rodi (1975), which is expected to capture the effects of distortion in the outer layer (Mastenbroek *et al.* 1996).

The above results do not imply that rapid-distortion turbulence models are physically less relevant in the theory of surface-wave generation. Although a mixing-length model proves to be appropriate in layers close to the water surface, the predicted relatively small stresses (compared with the 'classical' turbulence models) in the outer layer seem to agree qualitatively with measurements, see e.g. Mastenbroek *et al.* (1996).

A more rational conclusion is that the present theories for wave generation by turbulent air flow do not take into account an (as yet unknown) essential wave growth mechanism, which enhances the growth rate significantly. Apparently, the failure of classical turbulence models and their recent modifications to predict the rate of growth of the generated wave satisfactorily means that a different approach is needed to resolve this intriguing question.

Appendix. The solutions in the various layers

A.1. *The outer layer*

$$\bar{\theta}_0 = -wz, \quad w = 1-c, \quad \bar{\theta}_1 = \frac{z - z\log z}{\kappa}. \tag{A 1}$$

$$\theta_1 = w\,\mathrm{e}^{-z}, \quad \theta_2 = -\frac{1}{\kappa}E_1(2z)\,\mathrm{e}^{z} - \frac{1}{\kappa}(\gamma + \log 2)\,\mathrm{e}^{-z}, \tag{A 2}$$

$$\theta_{3r} = \mathrm{i}h_3\,\mathrm{e}^{-z} + \frac{2\mathrm{i}\kappa}{h^2}\left[\frac{-\mathrm{e}^{-z}}{z} + E_1(2z)\,\mathrm{e}^{z} + (\log z)\,\mathrm{e}^{-z}\right]\delta_{n1}. \tag{A 3}$$

A.2. *The transition layer*

$$\psi_1 = -w\zeta, \quad \psi_2 = \frac{w\zeta^2}{2}, \quad \psi_3 = \frac{\zeta}{2\kappa}, \quad \psi_5 = \frac{\zeta^2}{4\kappa}, \tag{A 4}$$

$$\psi_{4r} = \frac{-2\mathrm{i}\kappa\zeta}{1+(h\zeta)^{2n}}, \quad \psi_{6r} = \frac{2\mathrm{i}\kappa\zeta^2}{1+(h\zeta)^{2n}}\left(1 - \frac{1}{\kappa w\zeta^2}\right) + \mathrm{i}\alpha_6. \tag{A 5}$$

A.3. *The intermediate layer*

$$\varphi_2 = -w\eta, \quad \varphi_3 = \frac{\eta}{\kappa}, \quad \varphi_4 = \frac{w\eta^2}{2} + \frac{\eta\log\eta}{\kappa} + \alpha_1\eta + \alpha_2[-1+\omega_3(\eta)], \tag{A 6a–c}$$

where $\omega_3(\eta)$ satisfies the equation

$$\left(2\kappa\eta\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - \mathrm{i}w\right)\omega_3 = 0, \tag{A 7}$$

with

$$\omega_3(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \quad \omega_3(\eta) = 1 + \left(\frac{iw}{2\kappa}\right) \eta \log \eta + O(\eta) \quad \text{as} \quad \eta \to 0.$$

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